RELATIONS BETWEEN WORDS IN TWO POSITIVE DEFINITE MATRICES

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Abstract. A generalized word in two letters A and B is an expression of the form $W(A,B) = A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_k}B^{p_k}A^{p_{k+1}}$ in which p_i,q_i are real numbers such that $p_i, q_i \neq 0$, $i = 1,\ldots,k$, and p_{k+1} is arbitrary. We are interested when positive definite (complex Hermitian) matrices are substituted for A and B in the word $W(A,B)$. Specifically, it is shown that two non-identical generalized words cannot define the same function on the set of 2-by-2 positive definite matrices. A corollary is that a generalized word is positive definite for all positive definite A and B if and only if the word is symmetric ("palindromic"). This elaborates upon a remark made in a previous work by the author concerning positive definite word equations.

1. Introduction

A *generalized word* (*g-word*, for short) $W = W(A,B)$ in two letters A and B is an expression of the form $W = A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_k}B^{q_k}A^{p_{k+1}}$ in which the exponents p_i and q_i are real numbers such that $p_i, q_i \neq 0$, $i = 1,\ldots,k$, and p_{k+1} is an arbitrary real number. The *reversal* of the g-word W is $W^* = A^{p_{k+1}}B^{q_k}A^{p_k}\cdots B^{q_2}A^{p_2}B^{q_1}A^{p_1}$, and a g-word is *symmetric* if it is identical to its reversal (in other contexts, the name "palindromic" is also used). We will call a g-word, W, A-*positive* if all exponents of A in W are positive.

We are interested in the matrices that result when the two letters are positive definite (complex Hermitian) n-by-n matrices. To make sure that W is well-defined after substitution, we take primary powers (see [4, p. 433] and [4, p. 413]). That is, given $p \in \mathbb{R}\backslash\{0\}$, a unitary matrix U, and a nonnegative diagonal matrix D, we have $(UDU^*)^p = UD^pU^*$.

In [1], building on the work of [5], the authors study a certain type of matrix equation involving A-positive symmetric g-words.

Definition 1.1. A *symmetric word equation* is an equation, $S(A,B) = P$, in which $S(A,B)$ is an A-positive symmetric g-word. If B and P are given positive definite matrices, any positive definite matrix A for which the equation holds is called a *solution* to the symmetric word equation.

A symmetric word equation is called *solvable* if there exists a solution for every pair of positive definite n-by-n B,P. The main result of [1] is the following general fact.

Theorem 1.2. *Every symmetric word equation is solvable.*

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The purpose of this note is to explain the significance of the symmetric restriction in the definition of "symmetric word equation." Specifically, we prove that a generalized word is positive definite for all positive definite A and B if and only if the word is symmetric.

2. Relations between positive definite words

We begin by illustrating some of the subtlety of the problem. Let B and P be positive definite matrices. Then, it is known [5] that

$$P^{1/2}\left(P^{-1/2}BP^{-1/2}\right)^{1/2}P^{1/2} = B^{1/2}\left(B^{-1/2}PB^{-1/2}\right)^{1/2}B^{1/2},$$

even though both expressions appear to be quite different. In fact, both sides of the above equality are the unique solution A to the symmetric word equation,

$$S(A,B) = AB^{-1}A = P.$$

Fortunately, such behavior does not occur with g-words, as the following fact illustrates. The idea for the argument was inspired from a calculation made in [2].

Theorem 2.1. *A generalized word $W(A,B)$ is equal to the identity matrix for all substitutions of 2-by-2 positive definite A and B if and only if W is the empty word.*

Proof. Let $W = A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_k}B^{q_k}A^{p_{k+1}}$ in which p_i,q_i are real numbers such that $p_i, q_i \neq 0$, $i = 1,\ldots,k$. If $W = A^{p_1}$ ($k = 0$), then W is the identity if and only if $p_1 = 0$ or $A = I$ (by the uniqueness of taking positive definite p^{th} roots). Therefore, we may assume that $k \geq 1$. Furthermore, by performing a similarity using the last letter, we may also suppose that $W = A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_k}B^{q_k}$ in which p_i,$q_i \neq 0$.

We will show, by way of contradiction, that W cannot be the identity matrix for all 2-by-2 positive definite A and B. First, notice that at least one of the p_i must be negative since setting $B = I$ and $A \neq I$ gives a contradiction. Next, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}, \quad B = \begin{bmatrix} 1/2+\epsilon & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

for some $\epsilon > 0$. An easy computation shows that the matrix

$$2^{\sum_{q_j<0} q_j}\epsilon^{-(\sum_{p_j<0} p_j + \sum_{q_j<0} q_j)}A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_k}B^{q_k} \tag{2.1}$$

is the product of $2k$ matrices the $(2j-1)$-st of which is $\begin{bmatrix} 1 & 0 \\ 0 & \epsilon^{p_j} \end{bmatrix}$ if $p_j > 0$ or $\begin{bmatrix} \epsilon^{-p_j} & 0 \\ 0 & 1 \end{bmatrix}$ if $p_j < 0$, and the $2j$-th of which is $\begin{bmatrix} 1/2+\epsilon & 1/2 \\ 1/2 & 1/2 \end{bmatrix}^{q_j}$ if $q_j > 0$ or $\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2+\epsilon \end{bmatrix}^{-q_j}$ if $q_j < 0$, $j = 1,\ldots,k$.

Thus, the limit of (2.1) for $\epsilon \to 0$ exists and equals

$$P_1Q_1P_2Q_2\cdots P_kQ_k, \tag{2.2}$$

where P_j is $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ if $p_j > 0$ and $I-P$ if $p_j < 0$, and Q_j is $Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ if $q_j > 0$ and $I - Q$ if $q_j < 0$.

Assuming that the word W is the identity matrix for all A,B, it follows that for all $\epsilon > 0$, expression (2.1) is just $2^{\sum_{q_j<0} q_j}\epsilon^{-(\sum_{p_j<0} p_j+\sum_{q_j<0} q_j)}I$. Since the limit of (2.1) exists, it must agree with

$$\lim_{\epsilon\to 0} 2^{\sum_{q_j<0} q_j}\epsilon^{-(\sum_{p_j<0} p_j+\sum_{q_j<0} q_j)}I = 0$$

(since $-(\sum_{p_j<0} p_j + \sum_{q_j<0} q_j) > 0$). Finally, Lemma 2.2 below shows that (2.2) can never be zero, a contradiction that finishes the proof. $\square$

Lemma 2.2. *Let P_i, Q_i be letters ($i = 1, \ldots, k$), and let W be a word with alternating P_i's and Q_i's (e.g. $P_1Q_1P_2Q_2\cdots P_kQ_k$, $Q_1P_2Q_2\cdots P_kQ_k$). Then, for all substitutions of the P_i from the set $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ and the Q_i from the set $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right\}$, we never have $W = 0$.*

Proof. Let M be the matrix produced after substitution of the letters P_i, Q_i into a word W as in the statement of the lemma. We claim that $M \neq 0$. Indeed, suppose that $M = 0$; we will derive a contradiction. By multiplying (if necessary) M on the right by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we may assume that W ends in the letter P_k. Let $v = [x, y]^T$ and suppose that W begins with Q_1. Then, the only possible outcomes for Mv are: $[\pm x, \pm x]^T$, $[\pm x, \mp x]^T$, $[\pm y, \pm y]^T$, $[\pm y, \mp y]^T$. Similarly, if W begins with P_1, then Mv must be one of the following: $[\pm x, 0]^T$, $[0, \pm x]^T$, $[\pm y, 0]^T$, $[0, \pm y]^T$. These statements are easily proved by induction on the length of the word W. It is therefore clear that one can choose x and y such that $Mv \neq 0$. This contradiction completes the proof of the lemma. $\square$

We now list some corollaries to Theorem 2.1.

Corollary 2.3. *If two generalized words are equal for all 2-by-2 substitutions of positive definite A and B, then they are identical.*

Proof. Clear from Theorem 2.1. $\square$

Corollary 2.4. *The following are equivalent for a generalized word W.*

1. *W is positive definite for all substitutions of positive definite A and B*
2. *W is Hermitian for all substitutions of positive definite A and B*
3. *W is Hermitian for all 2-by-2 substitutions of positive definite A and B*
4. *W is symmetric ("palindromic")*

In particular, if a generalized word is Hermitian for all 2-by-2 substitutions of positive definite A and B, then the word is necessarily positive definite for all such substitutions.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear. If $W(A, B)$ is always Hermitian for 2-by-2 positive definite A and B, then $W(A, B)^* = W(A, B)$ for all such A and B. But then Corollary 2.3 says that W^* and W must be identical as words. It follows that W is symmetric. This proves (3) $\Rightarrow$ (4). Finally, if W is symmetric, an elementary congruence argument (see, for instance, [1] or [3, p. 223]) shows that W will always be positive definite for any positive definite A and B. This completes the proof. $\square$

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