11077: THE ASYMPTOTIC BEHAVIOR OF A CERTAIN SUM

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Abstract. [**11077**] Let $x_n = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \frac{2^{n-i}-1}{n-i}$. Find the asymptotic behavior of x_n.

1. solution

We prove that

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \frac{2^{n-i}-1}{n-i} = (-1)^n \sum_{j=1}^{n} \frac{(-1)^j}{j}.$$

This shows that x_n tends to $(-1)^n \ln 2$. First preprocess (by replacing $n-i$ with j) the given equation so that we need only show:

$$\sum_{j=1}^{n} (-1)^j \binom{n}{j} \frac{2^j-1}{j} = \sum_{j=1}^{n} \frac{(-1)^j}{j}.$$

Clearly, the relation holds for $n = 1$ as both sides are -1. Writing y_n for the right-hand-side (and z_n for the left-hand-side), notice that $y_n = y_{n-1} + (-1)^n/n$. Since z_1 and y_1 agree, it suffices to verify that z_n satisfies the same recurrence as y_n. Examine the difference

$$\begin{aligned}
z_{n+1} - z_n &= \sum_{j=1}^{n+1} (-1)^j \binom{n}{j-1} \frac{2^j-1}{j} \\
&= \frac{1}{n+1} \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} (2^j-1) \\
&= \frac{1}{n+1} \left[\sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} 2^j - \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} \right] \qquad (1.1) \\
&= \frac{1}{n+1} \left[(1-2)^{n+1} - 1 - ((1-1)^{n+1} - 1) \right] \\
&= \frac{1}{n+1} (-1)^{n+1}.
\end{aligned}$$

This completes the proof.

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