MATHEMATICAL MONTHLY PROBLEM SOLUTION

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Abstract. [11028]. Proposed by *Götz Trenkler.* Let P and Q be Hermitian idempotent n-by-n matrices with complex entries. That is, each is equal to its own square and its own conjugate transpose. Show that PQ shares these properties if and only if the trace of $PQPQ$ is equal to the trace of PQ.

1. Solution

One direction is obvious. As for the converse, suppose that $\mathrm{Tr}[PQPQ - PQ]$ is zero. Consider now $\mathrm{Tr}\left[(QP - PQ)(QP - PQ)^*\right]$, which is zero if and only if $QP - PQ = 0$ (since $\mathrm{Tr}[AA^*]$ is the sum of the squares of the absolute values of the entries of A).

An easy computation using the assumptions does indeed give us 0 for such a trace:

$$\begin{aligned}
\mathrm{Tr}\left[(QP - PQ)(QP - PQ)^*\right] &= \mathrm{Tr}[QP^2Q - QPQP - PQPQ + PQ^2P] \\
&= \mathrm{Tr}[QPQ] - \mathrm{Tr}[QPQP] - \mathrm{Tr}[PQPQ] + \mathrm{Tr}[PQP] \\
&= 2\mathrm{Tr}[PQ] - 2\mathrm{Tr}[PQPQ] \\
&= 0
\end{aligned}$$

by assumption and repeatedly using the fact that $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$. It follows, therefore, that

$$QP - PQ = 0$$

which solves the problem.

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