MARCH 2002 MATHEMATICAL MONTHLY PROBLEM

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ABSTRACT. [10928] Proposed by *Christopher J. Hillar*. Let a_n be a sequence of positive real numbers such that $\sum_{i=1}^{\infty} a_n$ converges. Then, (a) Prove that $\sum_{i=1}^{\infty} a_n^{\log n/(1+\log n)}$ converges. (b) Prove or disprove: If b_n is an increasing sequence of positive real numbers tending to 1, then $\sum_{i=1}^{\infty} a_n^{b_n}$ converges.

1. SOLUTION

(a) Let $f(n) = \frac{\ln n}{1+\ln n}$ and suppose $w \in \{1, 2, \ldots\}$. Examine the equality,

$$(wa_n)^{f(n)} \left(\left(\frac{1}{w} \right)^{\frac{f(n)}{1-f(n)}} \right)^{1-f(n)} = a_n^{f(n)} \tag{1.1}$$

Now, a form of Holder's Inequality states that if $x, y > 0$ and $a + b = 1$, then $x^a y^b \leq ax + by$. Hence, (1.1) gives us that

$$a_n^{f(n)} \leq wa_n f(n) + (1 - f(n)) \left(\frac{1}{w} \right)^{\frac{f(n)}{1-f(n)}} \tag{1.2}$$

Next, notice that

$$(1 - f(n)) \left(\frac{1}{w} \right)^{\frac{f(n)}{1-f(n)}} = \left(\frac{1}{1 + \ln n} \right) \left(\frac{1}{w} \right)^{\ln n} = \left(\frac{1}{1 + \ln n} \right) n^{\ln\left(w^{-1}\right)}$$

Now, choose w large enough so that $(\ln w^{-1}) < -1$ ($w = 3$ will do). Hence, both parts of right hand side of (1.2) have convergent sums. In fact,

$$\sum a_n^{f(n)} \leq w \sum a_n + \sum n^{\ln\left(w^{-1}\right)} < \infty$$

(b) We will show a counterexample with the sequence, $a_n = 1/(n(\ln n)^2)$ and the function, $g(n) = \frac{\ln \ln n}{1+\ln \ln n}$. Indeed, a simple calculation reveals that

$$\left(\frac{1}{n\,(\ln n)^2} \right)^{\frac{\ln \ln n}{1+\ln \ln n}} = \frac{1}{(\ln n)^{\frac{\ln n}{1+\ln \ln n}} (\ln n)^{\frac{2 \ln \ln n}{1+\ln \ln n}}} = \frac{1}{(\ln n)^{\frac{\ln n + 2 \ln \ln n}{1+\ln \ln n}}}$$

But for sufficiently large n (in fact, $n > 2^{128}$), we have

$$\frac{\ln n + 2 \ln \ln n}{1 + \ln \ln n} < \frac{\ln n}{\ln \ln n}$$

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Hence,

$$\frac{1}{(\ln n)^{\frac{\ln n + 2\ln\ln\ n}{1+\ln\ln\ n}}} \;\geq\; \frac{1}{(\ln n)^{\frac{\ln n}{\ln\ln\ n}}} \;=\; \frac{1}{n}$$

Therefore, $\sum {a_n}^{g(n)}$ diverges.