2001 MONTHLY PROBLEM

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ABSTRACT. [10851] Proposed by *David Beckwith, Sag Harbor, NY.* Prove that any gathering of people may be divided into reds and blues in such a way that each red is acquainted with an even number of reds and each blue is acquainted with an odd number of reds. (Assume that acquaintance is a symmetric and irreflexive relation).

1. SOLUTION

Proof. It is readily seen that this problem is equivalent to the following: for every simple graph, there is a coloring of the vertices into blue and red so that each blue vertex is connected to an odd number of red vertices and each red vertex is connected to an even number of red vertices.

Let G be a simple graph with n vertices, and let A be the adjacentcy matrix for G. Let $x = [x_1 \cdots x_n]^T \in \mathbb{Z}_2^n$ represent a coloring of the vertices in red and blue, where the ith vertex of G is colored red if $x_i = 1$ and blue if $x_i = 0$. It is clear that the ith vertex of G is connected to an even number of reds (under the coloring scheme provided by x) if the ith component $[Ax]_i$ of $Ax \in \mathbb{Z}_2^n$ is 0, and an odd number of reds if $[Ax]_i = 1$. Therefore we can color G appropriately if we can solve the matrix equation $Ax = 1 - x$, or $(I + A)x = 1$ over $\mathbb{Z}_2$.

We claim that for any symmetric n-by-n matrix $S = [s_{ij}]$ over $\mathbb{Z}_2$ with $s_{ii} = 1$, $i = 1, \ldots, n$, there is an $x \in \mathbb{Z}_2^n$ such that $Sx = 1$. (If we can show this, then the result follows since $I + A$ is a symmetric matrix over $\mathbb{Z}_2$ with ones along its main diagonal.) If S is nonsingular over $\mathbb{Z}_2$, then there is nothing to prove, so suppose S is singular. Choose a maximally linearly independent set L of rows of S, and put $R = \{m : s_{mj} \in L\}$. Since L is linearly independent, there is an $x \in \mathbb{Z}_2^n$ such that for each $s_{kj} \in L$, $\sum_{j=1}^n s_{kj}x_j = 1$. If $s_{\ell j}$ is any fixed row of S, then it is a linear combination of elements of L, and a linear combination over $\mathbb{Z}_2$ is simply a sum. Thus, for some $Q \subseteq R$, $s_{\ell j} = \sum_{m\in Q} s_{mj}$. Since S is symmetric,

$$\begin{aligned} 1 = s_{\ell\ell} &= \sum_{m\in Q} s_{m\ell} = \sum_{m\in Q} s_{\ell m} = \sum_{m,n\in Q} s_{nm} \\ &= \sum_{m\in Q} s_{mm} + 2 \sum_{\substack{m\neq n \\ n,m\in Q}} s_{nm} = \sum_{m\in Q} 1. \end{aligned}$$

Thus

$$\sum_{j=1}^n s_{\ell j}x_j = \sum_{j=1}^n x_j \sum_{m\in Q} s_{mj} = \sum_{m\in Q}\sum_{j=1}^n s_{mj}x_j = \sum_{m\in Q} 1 = 1.$$

Thus $Sx = 1$.

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If S is nonsingular, then there is exactly one solution. If S is singular, by doing Gaussian elimination on the linear independent set L, it becomes clear that the number of solutions x is always a power of 2.

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