SOLUTION TO PROBLEM 5.2 (C)

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ABSTRACT. We present a solution to the following problem: Let $a_1, \ldots, a_n \in \mathbb{C}$ and suppose that $S(k) = \sum_{i=1}^{n} {a_i}^k \in \mathbb{Z}$ for all $k \in \mathbb{P}$. Then,

$$\prod_{i=1}^{n} (t - a_i) \in \mathbb{Z}[t].$$

1. Newton Polynomials and Symmetric Functions

The problem above gives a converse to the following fact. Let α be an algebraic integer (a root of a monic polynomial, $g(t) \in \mathbb{Z}[t]$), then if $a_1, \ldots, a_n \in \mathbb{C}$ are the conjugates of this polynomial (all its roots), then $S(k) \in \mathbb{Z}$. This is an easier implication since the polynomial $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} {x_i}^k$ is symmetric and thus can be written (over $\mathbb{Z}$) in terms of the elementary symmetric polynomials. Substituting $(a_1, \ldots, a_n)$ for $(x_1, \ldots, x_n)$ in f gives us the result (as the coefficients of g are the elementary symmetric polynomials evaluated at $(a_1, \ldots, a_n)$).

We will first prove that $g(t) = \prod_{i=1}^{n} (t - a_i) \in \mathbb{Z}[t]$ is a polynomial in $\mathbb{Q}[t]$. This will follow from Newton's famous identities relating the coefficients of $g(t)$ to the values of $S(k)$.

Theorem 1.1. *Let $a_1, \ldots, a_n \in \mathbb{C}$ and let*

$$g(t) = \prod_{i=1}^{n} (t - a_i) = t^n + p_{n-1}x^{n-1} + \ldots + p_0.$$

Then,

$$S(k) + p_{n-1}S(k-1) + \ldots + p_{n-k+1}S(1) + kp_{n-k} = 0 \quad \textit{for} \quad k < n$$
$$S(k) + p_{n-1}S(k-1) + \ldots + p_1 S(k-n+1) + p_0 S(k-n) = 0 \quad \textit{for} \quad k \geq n$$

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